

# COCENTER OF $p$ -ADIC GROUPS, II: INDUCTION MAP

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**ABSTRACT.** In this paper, we study some relation between the cocenter  $\bar{H}(G)$  of the Hecke algebra  $H(G)$  of a connected reductive group  $G$  over a nonarchimedean local field and the cocenter  $\bar{H}(M)$  of its Levi subgroups  $M$ .

Given any Newton component of  $\bar{H}(G)$ , we construct the induction map  $\bar{i}$  from the corresponding Newton component of  $\bar{H}(M)$  to it. We show that this map is surjective. This leads to the Bernstein-Lusztig type presentation of the cocenter  $\bar{H}(G)$ , which generalizes the work [13] on the affine Hecke algebras. We also show that the map  $\bar{i}$  we constructed is adjoint to the Jacquet functor and in characteristic 0, the map  $\bar{i}$  is an isomorphism.

## INTRODUCTION

0.1. Let  $\mathbb{G}$  be a connected reductive group over a nonarchimedean local field  $F$  of arbitrary characteristic and  $G = \mathbb{G}(F)$ . Let  $R$  be an algebraically closed field of characteristic not equal to  $p$ , where  $p$  is the characteristic of residue field of  $F$ . Let  $H_R$  be the Hecke algebra of  $G$  over  $R$  and  $\bar{H}_R = H_R/[H_R, H_R]$  be its cocenter. Let  $\mathfrak{R}(G)_R$  be the  $R$ -vector space with basis the isomorphism classes of irreducible smooth admissible representations of  $G$  over  $R$ . Then we have the trace map

$$\mathrm{Tr}_R : \bar{H}_R \longrightarrow \mathfrak{R}(G)_R^*.$$

On the representation side, we have the induction functor and the Jacquet functor

$$i_{M,R} : \mathfrak{R}(M)_R \longrightarrow \mathfrak{R}(G)_R, \quad r_{M,R} : \mathfrak{R}(G)_R \longrightarrow \mathfrak{R}(M)_R,$$

where  $M$  is a Levi subgroup of  $G$ .

What happens on the cocenter side?

The functor adjoint to the induction functor  $i_M$  is the restriction map  $\bar{r}_{M,R} : \bar{H}(G)_R \rightarrow \bar{H}(M)_R$ . It can be expressed explicitly via the Van Dijk's formula. In this paper, we investigate the functor  $\bar{i}_{M,R} : \bar{H}_R(M) \rightarrow \bar{H}_R(G)$ , which is adjoint to the Jacquet functor  $r_{M,R} : \mathfrak{R}(G)_R \rightarrow \mathfrak{R}(M)_R$ .

0.2. We first describe the properties we expect for the map  $\bar{i}_{M,R}$  and then discuss the approach toward it.

First, instead of working over various algebraically closed fields  $R$ , it is desirable to have the map  $\bar{i}_M$  defined on the integral form  $\bar{H}$  (the cocenter of the Hecke algebra of  $\mathbb{Z}[\frac{1}{p}]$ -valued functions). Such map, if exists, provides not only a uniform approach to the map  $\bar{i}_{M,R}$  for all  $R$ , but also some useful information on the mod- $l$

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representations (see Theorem D in the introduction and a future work [6] for some results in this direction).

Second, in [11], we introduced the Newton decomposition. Roughly speaking,

$$G = \sqcup G(v) \quad \text{and} \quad \bar{H} = \oplus \bar{H}(v),$$

where  $v$  runs over the set of dominant rational coweights of  $G$ . Such description is expected to play an important role in the representation theory of  $p$ -adic groups. In order to relate the Newton decomposition with the representations, we would like to know that the Newton decomposition is compatible with the map  $\bar{i}_M$ .

0.3. Now we discuss several approaches in the literature towards the understanding of the map  $\bar{i}_M$ .

Over  $\mathbb{C}$ , the spectral density Theorem of Kazhdan [14] asserts that the trace map  $\text{Tr}_{\mathbb{C}} : \bar{H}_{\mathbb{C}} \rightarrow \mathfrak{R}(G)_{\mathbb{C}}^*$  is injective. Hence the map  $\bar{i}_{M,\mathbb{C}}$  is uniquely determined by the adjunction formula

$$\text{Tr}_{\mathbb{C}}^M(f, r_{M,\mathbb{C}}(\pi)) = \text{Tr}_{\mathbb{C}}^G(\bar{i}_{M,\mathbb{C}}(f), \pi).$$

However, if  $R$  is of positive characteristic, the trace map  $\text{Tr}_R$  may not be injective and thus the map  $\bar{i}_{M,R}$  is not uniquely determined by the adjunction formula.

In those cases, one may use the categorical description of the cocenter to give a definition of  $\bar{i}_{M,R}$ . Bernstein's second adjointness theorem implies that the map  $\bar{i}_{M,R}$  defined in this way is adjoint to the Jacquet functor (see [7, (1.8)]). However, it is not clear that this map preserves the integral structure (see some discussion in [7, §4.27]). Also it is not clear if this description is compatible with the Newton decomposition.

0.4. A different, but more explicit approach is given by Bushnell in [2].

Note that the induction functor  $i_{M,R}$  on the representations of  $M$  depends not only on the Levi subgroup  $M$ , but also on the parabolic subgroup  $P$  with Levi factor  $M$ . However, when passing to the Grothendieck group of the representations, the dependence of  $P$  disappears. On the other hand, the Jacquet functor  $r_{M,R}$ , even if one passes to the Grothendieck groups of the representations, still depend on the choice of parabolic subgroup.

Let  $v$  be a rational coweight. Then  $v$  determines a Levi subgroup  $M = M_v$  and the parabolic subgroup  $P_v = MN_v$ . Let  $\mathcal{K}$  be a “nice” open compact subgroup of  $G$  (e.g. the  $n$ -th congruent subgroup  $\mathcal{I}_n$  of an Iwahori subgroup) and  $\mathcal{K}_M = \mathcal{K} \cap M$ . Bushnell introduced the  $P_v$ -positive elements of  $M$  and the subalgebra  $H^v(M, \mathcal{K}_M)$  of  $H(M, \mathcal{K}_M)$ , consisting of compactly supported  $\mathcal{K}_M$ -biinvariant functions supported in the  $P_v$ -positive elements. Then he proves that

(a) The algebra  $H(M, \mathcal{K}_M)$  is isomorphic to the localization of  $H^v(M, \mathcal{K}_M)$  at a strongly positive element  $f_z$ .

(b) The map

$$j_{v,\mathcal{K}} : H^v(M, \mathcal{K}_M) \longrightarrow H(G, \mathcal{K}), \delta_{\mathcal{K}_M m \mathcal{K}_M} \longmapsto \delta_{P_v}(m)^{-\frac{1}{2}} \frac{\mu_G(\mathcal{K})}{\mu_M(\mathcal{K}_M)} \delta_{\mathcal{K} m \mathcal{K}}$$

is an injective algebra homomorphism.

(c) The map  $j_{v,\mathcal{K}}$  is adjoint to the Jacquet functor  $r_{M,\mathcal{K},R} : \mathfrak{R}_{\mathcal{K}}(G)_R \rightarrow \mathfrak{R}_{\mathcal{K} \cap M}(M)_R$  relative to  $P_v$ . Here  $\mathfrak{R}_{\mathcal{K}}(G)_R \subset \mathfrak{R}(G)_R$  consists of representations generated by their  $\mathcal{K}$ -fixed vectors.

Moreover, Bushnell's map  $j_{v,\mathcal{K}}$  also preserves the integral structure of the Hecke algebra.

0.5. It is tempting to apply Bushnell's result to the cocenter of Hecke algebras. However, there are several obstacles.

If  $\mathcal{K}$  is the Iwahori or pro- $p$  Iwahori subgroup, then the map  $j_{v,\mathcal{K}}$  extends to an algebra homomorphism  $H(M, \mathcal{K} \cap M) \rightarrow H(G, \mathcal{K})$ . In this case, the localization of Hecke algebra  $H^v(M, \mathcal{K} \cap M)$  is consistent with the Bernstein-Lusztig presentation ([10] and [18]). However, as pointed out in [2], these are essentially the only cases of this kind. Thus one may only use  $j_{v,\mathcal{K}}$  to deduce the induction map from part of the cocenter of  $H(M)$  to the cocenter of  $H(G)$ .

The Newton strata of  $M$  with integral dominant Newton points are positive, but the strata with rational (but not integral) Newton point may not be positive for any parabolic  $P$ . Those strata are not in the domain of the maps  $j_{v,\mathcal{K}}$ .

Also if one fixes  $M$  and  $P$ , the maps  $j_{v,\mathcal{K}}$  are not compatible with the change of open compact subgroups  $\mathcal{K}$ , even at the cocenter level (see §2.5). Thus the maps  $j_{v,\mathcal{K}}$  does not induce a well-defined map  $\bar{H}^v(M) \rightarrow \bar{H}$ .

0.6. The idea behind Bushnell's map  $j_{v,\mathcal{K}}$  is to enlarge the open compact subset  $\mathcal{K}_M m \mathcal{K}_M$  of  $M$  to the open compact subset  $\mathcal{K} m \mathcal{K}$  of  $G$  by multiplying the open compact subgroup  $\mathcal{K}$ . Inspired by it, we have the following construction.

Let  $v$  be a rational coweight and  $P = MN_v$  be the associated parabolic subgroup. The elements in the Newton stratum  $M(v)$  may not be  $P_v$ -positive, but a sufficiently large power of it is  $P_v$ -positive. One may enlarge an open compact subset inside  $M(v)$  by multiplying a suitable open compact subgroup of  $G$  to obtain an open compact subset of  $G$ . Unlike the situation in [2], the lack of  $P_v$ -positivity condition prevents us to give an explicit open compact subgroup of  $G$  that works in our situation. We have to use sufficiently small open compact subgroup of  $G$ . Since  $v$  is strictly positive with respect to  $N_v$ , we finally show that our construction is independent of the choice of such open compact subgroups. We have

**Theorem A.** *Let  $v$  be a rational coweight and  $M = M_v$ . Let  $\bar{v}$  be the  $G$ -dominant coweight associated to  $v$ . Then*

(1) [Theorem 3.1] *The map*

$$\delta_{m\mathcal{K}_M} \mapsto \delta_{P_v}(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M \mathcal{K})} \delta_{m\mathcal{K}_M \mathcal{K}} + [H, H]$$

*for sufficiently small open compact subgroup  $\mathcal{K}$  of  $G$  gives a well-defined map*

$$\bar{i}_v : \bar{H}(M; v) \longrightarrow \bar{H}.$$

(2) [Theorem 4.1] *The image of  $\bar{i}_v$  equals  $\bar{H}(G; \bar{v})$ .*

(3) [Theorem 6.5] *If moreover,  $\text{char}(F) = 0$ , then the map  $\bar{i}_v$  gives a bijection between  $\bar{H}(M; v)$  and  $\bar{H}(G; \bar{v})$ .*

**Theorem B** (Theorem 5.2). *Let  $v$  be a rational coweight and  $M = M_v$ . Then for any  $f \in \bar{H}_R(M; v)$  and  $\pi \in \mathfrak{R}(G)_R$ , we have the following adjunction formula*

$$\text{Tr}_R^M(f, r_{v,R}(\pi)) = \text{Tr}_R^G(\bar{i}_v(f), \pi).$$

*Here  $r_{v,R} : \mathfrak{R}(G)_R \rightarrow \mathfrak{R}(M)_R$  is the Jacquet functor relative to  $P_v$ .*

0.7. Now we discuss some applications. In [11], we introduced the rigid cocenter  $\bar{H}^{\text{rig}} = \oplus \bar{H}(v)$ , where  $v$  runs over rational central coweights.

Now for any standard Levi subgroup  $M$ , we introduce the  $+$ -rigid part  $\bar{H}(M)^{+, \text{rig}} = \oplus \bar{H}(M; v)$ , where  $v$  runs over rational dominant coweights with  $M = M_v$ . We then have the well-defined map

$$\bar{i}_M^+ = \oplus_v \bar{i}_v : \bar{H}(M)^{+, \text{rig}} \longrightarrow \bar{H}.$$

As an application of Theorem A and the Newton decomposition of  $\bar{H}$  (see [11, Theorem 3.1]), we have

**Theorem C.** *We have the decomposition of the cocenter  $\bar{H}$  into  $+$ -rigid parts:*

$$\bar{H} = \oplus_{M \text{ is a standard Levi subgroup}} \bar{i}_M^+ (\bar{H}(M)^{+, \text{rig}}).$$

For affine Hecke algebras, such decomposition is first obtained in [13] via an elaborate analysis on the minimal length elements in the affine Weyl groups of  $G$  and its Levi subgroups  $M$ . In loc.cit., such decomposition is called the Bernstein-Lusztig presentation of the cocenter of affine Hecke algebras, since the explicit expression of  $\bar{i}_M^+$  there is given in terms of the Bernstein-Lusztig presentation. Although there is no Bernstein-Lusztig type presentation for  $H$ , we follow [13] and still call the decomposition in Theorem C the Bernstein-Lusztig presentation of the cocenter  $\bar{H}$ . It is also worth mentioning that the proof in this paper does not involve the elaborate analysis on the minimal length elements as in [13], but based on the compatibility between the change of different open compact subgroups  $\mathcal{K}$  of  $G$ .

Theorem C asserts that the rigid cocenters of Levi subgroups form the “building blocks” of the whole cocenter  $\bar{H}$ . We also show that they are compatible with the trace map in the following way.

**Theorem D** (Theorem 6.1). *Let  $R$  be an algebraically closed field of characteristic not equal to  $p$ . Then we have*

$$\ker \text{Tr}_R = \oplus_{M \text{ is a standard Levi subgroup}} \bar{i}_M^+ (\ker \text{Tr}_R^M \cap \bar{H}_R(M)^{+, \text{rig}}).$$

If  $R = \mathbb{C}$ , we have the spectral density theorem and the kernel of the trace map is zero. Theorem D is trivial in this case. However, if  $R$  is of positive characteristic, especially when the spectral density theorem fails, then Theorem D would provide useful information toward the understanding of those representations.

0.8. The outline of the proof is as follows. In §2, we introduce the notion of quasi-positive elements and we use some remarkable properties on the minimal length elements established in [12] to show that any element in the Newton stratum  $M(v)$  is quasi-positive. Then in §3, we use the quasi-positivity to show that the map in Theorem A (1) is well-defined and factors through  $\bar{H}(M; v)$ . This proves part (1) of Theorem A.

As to part (2) of Theorem A, we first prove in Proposition 4.2 that  $M(v) \subset G(\bar{v})$ . Then by the admissibility of Newton strata ([11, Theorem 3.2]), any open compact subset  $X$  of  $M(v)$  enlarged by a sufficiently small open compact subgroup is still contained in  $G(\bar{v})$ . This shows that the image of  $\bar{i}_v$  is contained in  $\bar{H}(G; \bar{v})$ . The key ingredients in the proof of surjectivity are

- The notation of  $P$ -alcove elements introduced in [8].

- The Iwahori-Matsumoto presentation of  $\bar{H}(G; \bar{v})$  ([11, Theorem 4.1]).

By the quasi-positivity, for any  $f \in H(M; v)$ ,  $f^l \in H^v(M)$  for sufficiently large  $l$ . Theorem B follows from the adjunction formula proved in [2], the comparison between  $i_v(f)^l$  with  $j_{v,*}(f^l)$  and a trick of Casselman [4].

Finally, the injectivity in part (3) of Theorem A follows from the adjunction formula (Theorem B), the spectral density theorem and the freeness of the cocenter  $\bar{H}$  (which is only known in the case of  $\text{char}(F) = 0$ ).

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## 1. PRELIMINARY

1.1. Let  $\mathbb{G}$  be a connected reductive group over a nonarchimedean local field  $F$  of arbitrary characteristic. Let  $G = \mathbb{G}(F)$ . We fix a maximal  $F$ -split torus  $A$  and an alcove  $\mathfrak{a}_C$  in the corresponding apartment, and denote by  $\mathcal{I}$  the associated Iwahori subgroup.

Let  $Z = Z_G(A)$ . We denote by  $W_0 = N_G A(F)/Z(F)$  the *relative Weyl group* and  $\tilde{W} = N_G A(F)/Z_0$  the *Iwahori-Weyl group*, where  $Z_0$  is the unique parahoric subgroup of  $Z(F)$ .

We fix a special vertex of  $\mathfrak{a}_C$  and identify  $\tilde{W}$  as

$$\tilde{W} \cong X_*(Z)_{\text{Gal}(\bar{F}/F)} \rtimes W_0 = \{t^\lambda w; \lambda \in X_*(Z)_{\text{Gal}(\bar{F}/F)}, w \in W_0\}.$$

We have a semidirect product

$$\tilde{W} = W_a \rtimes \Omega,$$

where  $W_a$  is the affine Weyl group associated to  $\tilde{W}$  and  $\Omega$  is the stabilizer of the alcove  $\mathfrak{a}_C$  in  $\tilde{W}$ . Let  $\tilde{S}$  be the set of affine simple reflections of  $W_a$  determined by the fundamental alcove  $\mathfrak{a}_C$ . The groups  $W_a$  and  $\tilde{W}$  are equipped with a Bruhat order  $\leq$  and a length function  $\ell$ . The subgroup  $\Omega$  of  $\tilde{W}$  is the subgroup consisting of length-zero elements.

1.2. For any  $K \subset \tilde{S}$ , let  $W_K$  be the subgroup of  $\tilde{W}$  generated by  $s \in K$ . Let  ${}^K \tilde{W}$  be the set of elements  $w \in \tilde{W}$  of minimal length in the cosets  $W_K w$ .

Let  $\Phi = \Phi(G, A)$  be the set of roots of  $G$  relative to  $A$  and  $\Phi^+$  be the set of positive roots so that  $\mathfrak{a}_C$  is contained in the antidominant chamber of  $V$  determined by  $\Phi^+$ . Let  $\mathcal{R} = \{\alpha\}$  be the set of affine roots on  $\mathcal{A}$ . We choose a normalization of the valuation on  $F$  so that if  $\alpha \in \mathcal{R}$ , then so is  $\alpha \pm 1$  (see [1, §5.2.23]). For any  $n \in \mathbb{N}$ , let  $\mathcal{I}_n$  be the  $n$ -th Moy-Prasad subgroup associated to the barycenter of  $\mathfrak{a}_C$  [15]. This is the subgroup of  $G$  generated by the  $n$ -th congruence subgroup of  $Z(F)$  and the affine root subgroup  $X_{\alpha+n}$  for  $\alpha \in \mathcal{R}_+$ .

For any  $n \in \mathbb{N}$  and a subgroup  $G'$  of  $G$ , we set  $G'_n = G' \cap \mathcal{I}_n$ . We write  $\mathcal{I}_{G'}$  for  $G' \cap \mathcal{I}$ .

1.3. Let  $\mu_G$  be the Haar measure on  $G$  such that the pro- $p$  Iwahori subgroup  $\mathcal{I}'$  has volume 1. As in [11, Section 1], we denote by  $H = H(G)$  the Hecke algebra of locally constant, compactly supported  $\mathbb{Z}[\frac{1}{p}]$ -valued functions on  $G$ . We have

$$H = \varinjlim_{\mathcal{K}} H(G, \mathcal{K}),$$

where  $\mathcal{K}$  runs over open compact subgroups of  $G$  and  $H(G, \mathcal{K})$  is the space of compactly supported,  $\mathcal{K} \times \mathcal{K}$ -invariant  $\mathbb{Z}[\frac{1}{p}]$ -valued functions on  $G$ , i.e.,  $H(G, \mathcal{K}) = \bigoplus_{g \in \mathcal{K} \backslash G / \mathcal{K}} \mathbb{Z}[\frac{1}{p}] \delta_{\mathcal{K}g\mathcal{K}}$ , where  $\delta_{\mathcal{K}g\mathcal{K}}$  is the characteristic function on  $\mathcal{K}g\mathcal{K}$ .

We define the action of  $G$  on  $H$  by  ${}^x f(g) = f(x^{-1}gx)$  for  $f \in H$ ,  $x, g \in G$ . By [11, Proposition 1.1], the commutator  $[H, H]$  of  $H$  is the  $\mathbb{Z}[\frac{1}{p}]$ -submodule of  $H$  spanned by  $f - {}^x f$  for  $f \in H$  and  $x \in G$ . Let  $\bar{H} = H/[H, H]$  be the cocenter of  $H$ .

1.4. Now we recall the Newton decomposition introduced in [11].

Set  $V = X_*(Z)_{\text{Gal}(\bar{F}/F)} \otimes \mathbb{R}$  and  $V_+$  be the set of dominant elements in  $V$ . For any  $w \in \tilde{W}$ , there exists a positive integer  $l$  such that  $w^l = t^\lambda$  for some  $\lambda \in X_*(Z)_{\text{Gal}(\bar{F}/F)}$ . We set  $\nu_w = \lambda/l \in V$  and  $\bar{\nu}_w$  to be the unique dominant in the  $W_0$ -orbit of  $\nu_w$ . The element  $\nu_w$  and  $\bar{\nu}_w$  are independent of the choice of  $l$ .

Let  $\aleph = \Omega \times V_+$ . We have a map (see [11, §2.1])

$$\pi = (\kappa, \bar{\nu}) : \tilde{W} \longrightarrow \aleph, \quad w \longmapsto (wW_a, \bar{\nu}_w).$$

We denote by  $\tilde{W}_{\min}$  be the subset of  $\tilde{W}$  consisting of elements of minimal length in their conjugacy classes. For any  $\nu \in \aleph$ , we set

$$X_\nu = \bigcup_{w \in \tilde{W}_{\min}; \pi(w) = \nu} \mathcal{I} w \mathcal{I} \quad \text{and} \quad G(\nu) = G \cdot_\theta X_\nu.$$

Here  $\cdot$  means the conjugation action of  $G$ . Let  $H(\nu)$  be the submodule of  $H$  consisting of functions supported in  $G(\nu)$  and let  $\bar{H}(\nu)$  be the image of  $H(\nu)$  in the cocenter  $\bar{H}$ . The Newton decomposition of  $\bar{H}$  is established in [11, Theorem 3.1 (2)].

**Theorem 1.1.** *We have that*

$$\bar{H} = \bigoplus_{\nu \in \aleph} \bar{H}(\nu).$$

In this paper, we are mainly interested in the  $V$ -factor of  $\aleph$ . For any  $v \in V_+$ , we also set  $G(v) = \bigsqcup_{\nu=(\tau, v)} G(\nu)$  for some  $\tau \in \Omega$ ,  $H(v) = \bigoplus_{\nu=(\tau, v)} H(\nu)$  and  $\bar{H}(v) = \bigoplus_{\nu=(\tau, v)} \bar{H}(\nu)$ .

1.5. Let  $M$  be a semistandard Levi subgroup of  $G$ , i.e., a Levi subgroup of some parabolic subgroup of  $G$  that contains  $Z$ . Let  $\mathcal{I}_M = \mathcal{I} \cap M$  be the Iwahori subgroup of  $M$  and  $\tilde{W}(M)$  be the Iwahori-Weyl group of  $M$ . We denote by  $\tilde{S}(M)$  the set of affine simple reflections of  $\tilde{W}(M)$  determined by the Iwahori subgroup  $\mathcal{I}_M$ .

We may regard  $\tilde{W}(M)$  as a subgroup of  $\tilde{W}$  in a natural way. However, the length function  $\ell_M$  on  $\tilde{W}(M)$  does not equal to the restriction of  $\tilde{W}$  of the length function  $\ell$  on  $\tilde{W}$ .

Let  $\Omega_M$  be the subgroup of  $\tilde{W}(M)$  consisting of length-zero elements with respect to the length function  $\ell_M$ . We have  $\Omega_M \cong \tilde{W}(M)/W_a(M)$ , where  $W_a(M)$  is the affine Weyl group of the subgroup of  $\tilde{W}(M)$ . We have  $W_a(M) \subset W_a$  and



thus a natural map  $\Omega_M \cong \tilde{W}(M)/W_a(M) \rightarrow \tilde{W}/W_a \cong \Omega$ . Let  $V_+^M$  be the set of  $M$ -dominant elements in  $V$ . We set  $\aleph_M = \Omega_M \times V_+^M$  and we have a map  $\pi_M = (\kappa_M, \bar{\nu}_M) : \tilde{W}(M) \rightarrow \aleph_M$ .

We also have a natural map  $\aleph_M \rightarrow \aleph$  sending  $(\tau, v)$  to  $(\tau', \bar{v})$ , where  $\tau'$  is the image of  $\tau$  in  $\Omega$  and  $\bar{v}$  is the unique  $(G)$ -dominant element in the  $W_0$ -orbit of  $v$ .

Let  $\mu_M$  be the Haar measure on  $M$  such that the pro- $p$  Iwahori subgroup of  $M$  has volume 1. Let  $H(M)$  be the Hecke algebra of  $M$  and  $\bar{H}(M)$  be its cocenter. For any  $\nu_M \in \aleph_M$ , we denote by  $\bar{H}(M; \nu_M)$  the corresponding Newton component of  $\bar{H}(M)$ . By Theorem 1.1, we have

$$\bar{H}(M) = \oplus_{\nu_M \in \aleph_M} \bar{H}(M; \nu_M).$$

## 2. QUASI-POSITIVE ELEMENTS

2.1. The semistandard Levi may be described as the centralizer of elements in  $V$ . For any  $v \in V$ , we set  $\Phi_{v,0} = \{a \in \Phi; \langle a, v \rangle = 0\}$  and  $\Phi_{v,+} = \{a \in \Phi; \langle a, v \rangle > 0\}$ . Let  $M_v \subset G$  be the Levi subgroup generated by  $Z$  and  $U_a(F)$  for  $a \in \Phi_{v,0}$  and  $N_v \subset G$  be the unipotent subgroup generated by  $U_a(F)$  for  $a \in \Phi_{v,+}$ . Set  $P_v = M_v N_v$ . Then  $P_v$  is a semistandard parabolic subgroup and  $M_v$  is a Levi subgroup of  $P_v$ . We denote by  $P_v^- = M_v N_v^-$  the opposite parabolic. Let  $\mu_{N_v}, \mu_{N_v^-}$  be the Haar measures on  $N_v$  and  $N_v^-$  respectively such that  $\mu_G(nmn^-) = \mu_{N_v}(n)\mu_{M_v}(m)\mu_{N_v^-}(n^-)$  for  $n \in N_v, m \in M_v, n^- \in N_v^-$ . For  $m \in M_v$ , set  $\delta_v(m) = \frac{\mu_{N_v}(mN_{v,0}m^{-1})}{\mu_{N_v}(N_{v,0})}$ . For  $\nu = (\tau, v) \in \aleph$ , we may also write  $M_\nu$  for  $M_v$ ,  $N(\nu)$  for  $N_v$  and  $N^-(\nu)$  for  $N_v^-$ .

If  $v$  is dominant, then  $P_v$  is a standard parabolic subgroup of  $G$  and  $M_v$  is a standard Levi subgroup of  $G$ .

2.2. Let  $v \in V$ . Following [3, Definition 6.5 & Definition 6.14], we call an element  $m \in M_v$  a  $(P_v, \mathcal{I}_n)$ -positive element if

$$mN_{v,n}m^{-1} \subset N_{v,n}, \text{ and } m^{-1}N_{v,n}^-m \subset N_{v,n}^-.$$

We call an element  $z$  in the center of  $M_v$  a *strongly*  $P_v$ -positive element if the sequences  $z^n N_{v,0} z^{-n}, z^{-n} N_{v,0}^- z^n$  both tend monotonically to 1 as  $n \rightarrow \infty$ .

Following [2, §3.1], let  $H^v(M_v, M_{v,n})$  be the subalgebra of  $H(M_v, M_{v,n})$  of functions with support consisting of  $(P_v, \mathcal{I}_n)$ -positive elements. The following result is proved in [2, Proposition 5].

**Proposition 2.1.** *The map  $\delta_{M_{v,n}mM_{v,n}} \mapsto \delta_v(m)^{-\frac{1}{2}} \frac{\mu_{M_v}(M_{v,n})}{\mu_G(\mathcal{I}_n)} \delta_{\mathcal{I}_n m \mathcal{I}_n}$  defines an injective algebra homomorphism*

$$j_{v,n} : H^v(M_v, M_{v,n}) \hookrightarrow H(G, \mathcal{I}_n).$$

The formula we have here differs from [2] by the factor  $\delta_v(m)^{-\frac{1}{2}}$ , since in [2] the map is adjoint to the (unnormalized) Jacquet functor while we consider the (normalized) Jacquet functor.

By [2, §3.2],  $H(M_v, M_{v,n}) = S^{-1}H^v(M_v, M_{v,n})$  is the localization of  $H^v(M_v, M_{v,n})$ , where  $S = \langle \delta_{M_{v,n}zM_{v,n}} \rangle$  is the multiplicative closed set of the function  $\delta_{M_{v,n}zM_{v,n}}$  with a strongly  $P_v$ -positive element  $z$ . It is pointed out in [2, Remark 5] that the map  $j_{v,n}$  does not extend to an algebra homomorphism  $H(M_v, M_{v,n}) \rightarrow H(G, \mathcal{I}_n)$  for  $n > 0$ .

2.3. Let  $v \in V$  be a rational coweight and  $M = M_v$ . For any  $l \in \mathbb{N}$  with  $lv \in X_*(Z)$ , the element  $t^{lv}$  is strongly  $P_v$ -positive. However, in general, the element in  $M(v)$  may not be  $(P_v, *)$ -positive. Therefore, one can not deduce a map from  $\tilde{H}(M; v)$  to  $\tilde{H}$  via the map  $j_{v,n}$ .

**Example 2.2.** Let  $G$  be split  $GL_5$  and  $M = GL_3 \times GL_2$ . Let  $v = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{2}, \frac{1}{2})$ . Then  $M = M_v$ . The element  $w = t^{(1,1,0,1,0)}(132)(45)$  of  $\tilde{W}$  has Newton point  $v$ . However,  $w(e_4 - e_3) = e_5 - e_2 - 1$  is a negative affine root. Therefore the element  $\dot{w}$  is not  $(P_v, *)$ -positive.

2.4. To overcome the difficulty, we introduce the quasi-positive elements.

An element  $m \in M_v$  is called  $P_v$  quasi-positive if there exists  $l \in \mathbb{N}$  such that

$$(a) \quad m^l N_{v,n} m^{-l} \subset N_{v,n+1}, \text{ and } m^{-l} N_{v,n}^- m^l \subset N_{v,n+1}^- \text{ for any } n \in \mathbb{N}.$$

For any  $n \in \mathbb{N}$ ,  $w \in \tilde{W}$  and  $g \in \mathcal{I}\dot{w}\mathcal{I}$ , we have  $g\mathcal{I}_{n+\ell(w)}g^{-1} \subset \mathcal{I}_n$ . So

(b) Let  $w \in \tilde{W}(M)$  and  $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M$ . If  $m$  satisfies (a), then we have

$$m^n N_{v,n'+(l-1)\ell(w)} m^{-n} \subset N_{v,n'}, \text{ and } m^{-n} N_{v,n'+(l-1)\ell(w)}^- m^n \subset N_{v,n'}^- \text{ for any } n, n' \in \mathbb{N}.$$

We first discuss some properties on the quasi-positive elements.

**Proposition 2.3.** Let  $v \in V$  and  $M = M_v$ . Let  $w \in \tilde{W}(M)$  and  $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M$ . Suppose that  $m$  satisfying the inclusion relation §2.4 (a).

(1) For any  $n \in \mathbb{N}$ , any element in  $m\mathcal{I}_{n+(l-1)\ell(w)}$  is conjugate by an element in  $\mathcal{I}_n$  to an element in  $mM_{n+(l-1)\ell(w)}$ .

(2) For any  $n, n' \in \mathbb{N}$  and  $g \in \mathcal{I}_{n+(l-1)\ell(w)}$ , we have

$$\delta_{mgM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}} \equiv \delta_{mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}} \pmod{[H, H]}.$$

*Proof.* (1) We first show that

(a) For any  $i \in \mathbb{N}$ , any element in  $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$  is conjugate by  $\mathcal{I}_{n+i}$  to an element in  $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i+1}$ .

Note that any element in  $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$  is conjugate by  $\mathcal{I}_{n+(l-1)\ell(w)+i}$  to an element of the form  $u'gu$  with  $u' \in N_{v,n+(l-1)\ell(w)+i}^-$ ,  $g \in mM_{n+(l-1)\ell(w)}$  and  $u \in N_{v,n+(l-1)\ell(w)+i}$ . By §2.4 (b),  $gug^{-1} \in N_{v,n+i}$ . We have  $(u', gug^{-1}) \in (\mathcal{I}_{n+(l-1)\ell(w)+i}, \mathcal{I}_{n+i}) \subset \mathcal{I}_{n+(l-1)\ell(w)+i+1}$ . Now we have

$$u'gu = u'(gug^{-1})g \in (gug^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g.$$

So  $u'gu$  is conjugate by  $\mathcal{I}_{n+i}$  to an element in

$$\begin{aligned} u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g(gug^{-1}) &= u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}(g^2u(g^2)^{-1})g \\ &= u'(g^2u(g^2)^{-1})\mathcal{I}_{n+(l-1)\ell(w)+i+1}g \\ &= (g^2u(g^2)^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g. \end{aligned}$$

By the same procedure, for any  $l \in \mathbb{N}$ ,  $u'gu$  is conjugate by  $\mathcal{I}_{n+i}$  to an element in  $(g^l u(g^l)^{-1})u'\mathcal{I}_{n+(l-1)\ell(w)+i+1}g$ . By §2.4 (a),  $g^l u g^{-l} \in \mathcal{I}_{n+(l-1)\ell(w)+i+1}$ . Hence  $u'gu$  is conjugate by  $\mathcal{I}_{n+i}$  to an element in  $u'g\mathcal{I}_{n+(l-1)\ell(w)+i+1}g = u'g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$ . By the same argument, any element in  $u'g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$  is conjugate by  $\mathcal{I}_{n+i}$  to an element in  $g\mathcal{I}_{n+(l-1)\ell(w)+i+1}$ .

(a) is proved.

Let  $g_0 \in mM_n\mathcal{I}_{n+(l-1)\ell(w)}$ . By (a), we may construct inductively an element  $z_i \in \mathcal{I}_{n+i}$  for  $i \in \mathbb{N}$  such that  $g_{i+1} := z_i^{-1}g_iz_i$  is contained in  $mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+i}$ .



The convergent product  $z := z_1 z_2 \cdots$  is a well-defined element in  $\mathcal{I}_n$  and  $z^{-1}gz \in mM_{n+(l-1)\ell(w)}$ .

(2) By part (1), there exists  $h \in \mathcal{I}_{n+n'}$  such that  $hmgh^{-1} \in mM_{n+(l-1)\ell(w)}$ . We have  $(\mathcal{I}_{n+n'}, M_{n+(l-1)\ell(w)}) \subset \mathcal{I}_{n+n'+(l-1)\ell(w)}$ . Therefore  $M_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}$  is a subgroup of  $\mathcal{I}$  and is stable under the conjugation action of  $\mathcal{I}_{n+n'}$ . Thus  $hmgM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}h^{-1} = mM_{n+(l-1)\ell(w)}\mathcal{I}_{n+(l-1)\ell(w)+n'}$ . The statement is proved.  $\square$

2.5. We say that  $m \in M$  is  $P_v$  strictly positive if for any  $n \in \mathbb{N}$ , we have

$$mN_{v,n}m^{-1} \subset N_{v,n+1}, \text{ and } m^{-1}N_{v,n}^-m \subset N_{v,n+1}^-.$$

We denote by  $H^{v^\sharp}(M)$  the subalgebra of  $H(M)$  consisting of functions with support consisting of  $P_v$  strictly positive elements. Note that the limit of the support of  $j_{v,n}(\delta_{Z_0})$  for  $v$  dominant regular, as  $n$  goes to infinite, is just  $Z_0$  itself, but the support of  $j_{v,n}(\delta_{Z_0})$  for each  $n$  contains of nonsplit regular semisimple elements. Thus the maps  $\{j_{v,n}\}$  are not compatible with the natural maps  $\bar{H}^v(M, M_n) \rightarrow \bar{H}^v(M, M_{n+1})$ .

However, we have the following compatibility result for  $P_v$  strictly positive part.

**Corollary 2.4.** *Let  $n \in \mathbb{N}$ . Then the following diagram commutes*

$$\begin{array}{ccc} \bar{H}^{v^\sharp}(M, M_n) & \xrightarrow{j_{v,n}} & \bar{H}(G, \mathcal{I}_n) \\ \downarrow & & \downarrow \\ \bar{H}^{v^\sharp}(M, M_{n+1}) & \xrightarrow{j_{v,n+1}} & \bar{H}(G, \mathcal{I}_{n+1}). \end{array}$$

*Proof.* Let  $m \in M$  be  $P_v$  strictly positive. Then  $\delta_{M_n m M_n} \in H^{v^\sharp}(M, M_n) \subset H^{v^\sharp}(M, M_{n+1})$ . By definition,

$$j_{v,n+1}(\delta_{M_n m M_n}) = \delta_v(m)^{-\frac{1}{2}} \frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})} \delta_{\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1}}.$$

Note that  $\mathcal{I}_{n+1} M_n = M_n \mathcal{I}_{n+1}$  is a subgroup of  $\mathcal{I}$ . We have

$$\mathcal{I}_n m \mathcal{I}_n = \sqcup_{(i_1, i_2, i'_1, i'_2)} i_1 i'_1 \mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1} i'_2 i_2,$$

where  $\{(i_1, i_2, i'_1, i'_2)\} \subset N_n \times N_n \times N_n^- \times N_n^-$  is a finite subset. By Proposition 2.3 (2), for  $i_1, i_2 \in N_n$  and  $i'_1, i'_2 \in N_n^-$ , we have

$$\delta_{i_1 i'_1 \mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1} i'_2 i_2} \equiv \delta_{\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1}} \pmod{[H, H]}.$$

Thus

$$j_{v,n}(\delta_{M_n m M_n}) \equiv \delta_v(m)^{-\frac{1}{2}} \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})} \delta_{\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1}} \pmod{[H, H]}.$$

It remains to show that  $\frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})} = \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})}$ .

Suppose that  $m \in \mathcal{I}_M w \mathcal{I}_M$  for some  $w \in \tilde{W}(M)$ . By [11, Lemma 4.6],

$$\frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_n)} = \frac{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})}{\mu_G(\mathcal{I}_{n+1})} = q^{\ell(w)}, \frac{\mu_M(M_n m M_n)}{\mu_M(M_n)} = \frac{\mu_M(M_{n+1} m M_{n+1})}{\mu_M(M_{n+1})} = q^{\ell_M(w)}.$$

Now we have

$$\begin{aligned}
\frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})} &= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})} \frac{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})}{\mu_G(\mathcal{I}_{n+1} M_n m M_n \mathcal{I}_{n+1})} \\
&= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n m \mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1} m \mathcal{I}_{n+1})} \frac{\mu_M(M_{n+1} m M_{n+1})}{\mu_M(M_n m M_n)} \\
&= \frac{\mu_M(M_n)}{\mu_G(\mathcal{I}_n)} \frac{\mu_G(\mathcal{I}_n)}{\mu_G(\mathcal{I}_{n+1})} \frac{\mu_M(M_{n+1})}{\mu_M(M_n)} = \frac{\mu_M(M_{n+1})}{\mu_G(\mathcal{I}_{n+1})}.
\end{aligned}$$

The statement is proved.  $\square$

Finally we show that the elements in  $M_\nu(\nu)$  are  $P_\nu$  quasi-positive.

**Proposition 2.5.** *Let  $v \in V$  be a rational coweight and  $M = M_v$ . Let  $w \in \tilde{W}(M)$ . Then there exists a positive integer  $i_{v,w}$  such that for any  $m \in \mathcal{I}_M \dot{w} \mathcal{I}_M \cap M(v)$  and  $n \geq i_{v,w}$ , we have*

$$m^{i_{v,w}} N_{v,n} (m^{i_{v,w}})^{-1} \subset N_{v,n+1}, \quad (m^{i_{v,w}})^{-1} N_{v,n}^- m^{i_{v,w}} \subset N_{v,n+1}^-.$$

2.6. The proof relies on some remarkable properties of the Iwahori-Weyl group, which we recall here.

For  $w, w' \in \tilde{W}$  and  $s \in \tilde{\mathbb{S}}$ , we write  $w \xrightarrow{s} w'$  if  $w' = sws$  and  $\ell(w') \leq \ell(w)$ . We write  $w \rightarrow w'$  if there is a sequence  $w = w_0, w_1, \dots, w_n = w'$  of elements in  $\tilde{W}$  such that for any  $1 \leq k \leq n$ ,  $w_{k-1} \xrightarrow{s_k} w_k$  for some  $s_k \in \tilde{\mathbb{S}}$ . We write  $w \approx w'$  if  $w \rightarrow w'$  and  $w' \rightarrow w$ . It is easy to see that if  $w \rightarrow w'$  and  $\ell(w) = \ell(w')$ , then  $w \approx w'$ . We have that

(a) If  $w \xrightarrow{s} w'$  and  $\ell(w) = \ell(w')$ , then for any  $g \in \mathcal{I} \dot{w} \mathcal{I}$ , there exists  $g' \in \mathcal{I} \dot{s} \mathcal{I}$  such that  $g' g (g')^{-1} \in \mathcal{I} \dot{w}' \mathcal{I}$ .

(b) If  $w \xrightarrow{s} w'$  and  $\ell(w') < \ell(w)$ , then for any  $g \in \mathcal{I} \dot{w} \mathcal{I}$ , there exists  $g' \in \mathcal{I} \dot{s} \mathcal{I}$  such that  $g' g (g')^{-1} \in \mathcal{I} \dot{w}' \mathcal{I} \sqcup \mathcal{I} \dot{s} \dot{w} \mathcal{I}$ .

An element  $w \in \tilde{W}$  is called *straight* if  $\ell(w^n) = n\ell(w)$  for any  $n \in \mathbb{N}$ . A triple  $(x, K, u)$  is called a *standard triple* if  $x \in \tilde{W}$  is straight,  $K \subset \tilde{\mathbb{S}}$  with  $W_K$  finite,  $x \in {}^K \tilde{W}$  and  $\text{Ad}(x)(K) = K$ , and  $u \in W_K$ . By definition,

(c) For any  $n \in \mathbb{N}$  and  $g_1, \dots, g_n \in \mathcal{I} \dot{x} \mathcal{I}$ , we have  $g_1 g_2 \dots g_n \in (\mathcal{I} W_K \mathcal{I})(\mathcal{I} \dot{x}^n \mathcal{I})$ .

It is proved in [12, Theorem A & Proposition 2.7] that

**Theorem 2.6.** *For any  $w \in \tilde{W}$ , there exists a standard triple  $(x, K, u)$  such that  $ux \in \tilde{W}_{\min}$  and  $w \rightarrow ux$ . In this case,  $\pi(w) = \pi(x)$ .*

Following [9, §4.3], we write  $w \xrightarrow{s} w'$  if either  $w \xrightarrow{s} w'$  or  $w' = sw$  and  $\ell(w) > \ell(sws)$ , and we write  $w \rightharpoonup w'$  if there exists a sequence  $w = w_0, w_1, \dots, w_n = w'$  of elements in  $\tilde{W}$  such that for any  $1 \leq k \leq n$ ,  $w_{k-1} \xrightarrow{s_k} w_k$  for some  $s_k \in \tilde{\mathbb{S}}$ . It is easy to see that if  $w \in \tilde{W}_{\min}$  and  $w \rightharpoonup w'$ , then  $w \approx w'$ .

We show that

**Lemma 2.7.** *Let  $w \in \tilde{W}$  and  $g \in \mathcal{I} \dot{w} \mathcal{I}$ . Then there exists a standard triple  $(x, K, u)$ , a sequence  $w = w_0, w_1, \dots, w_n = ux$  of distinct elements in  $\tilde{W}$  and a sequence  $g = g_0, g_1, \dots, g_n$  of elements in  $G$  such that*

- (1)  $ux \in \tilde{W}_{\min}$ ;
- (2) for any  $0 \leq k \leq n$ ,  $g_k \in \mathcal{I} \dot{w}_k \mathcal{I}$ ;

(3) for any  $1 \leq k \leq n$ , there exists  $s_k \in \tilde{\mathbb{S}}$  and  $h_k \in \mathcal{I}\dot{s}_k\mathcal{I}$  such that  $w_{k-1} \xrightarrow{s_k} w_k$  and  $g_k = h_k g_{k-1} h_k^{-1}$ .

**Remark 2.8.** By definition, if  $w \rightarrow w'$ , then  $w' \in wW_a$  and  $\ell(w') \leq \ell(w)$ . In particular, the length of the sequence is at most  $\#\{x \in W_a; \ell(x) \leq \ell(w)\}$ .

*Proof.* We argue by induction on  $\ell(w)$ .

If  $w \in \tilde{W}_{\min}$ , by Theorem 2.6, there exists a standard triple  $(x, K, u)$  with  $ux \in \tilde{W}_{\min}$  and a sequence  $w = w_0, w_1, \dots, w_n = ux$  of distinct elements in  $\tilde{W}$  such that for any  $1 \leq k \leq n$ ,  $w_{k-1} \xrightarrow{s_k} w_k$  for some  $s_k \in \tilde{\mathbb{S}}$ . Since  $w \in \tilde{W}_{\min}$ , we have  $\ell(w_k) = \ell(w)$  for all  $k$ . Now the statement follows from §2.6 (a).

If  $w \notin \tilde{W}_{\min}$ , then by Theorem 2.6, there exists a sequence  $w = w_0, w_1, \dots, w_n$  of distinct elements in  $\tilde{W}$  such that  $\ell(w) = \ell(w_n)$ , for any  $1 \leq k \leq n$ ,  $w_{k-1} \xrightarrow{s_k} w_k$  for some  $s_k \in \tilde{\mathbb{S}}$  and there exists  $s \in \tilde{\mathbb{S}}$  with  $sw_n s < w_n$ . Then we have  $\ell(w_k) = \ell(w)$  for all  $k$ . By §2.6 (a), for any  $1 \leq k \leq n$ , there exists  $h_k \in \mathcal{I}\dot{s}_k\mathcal{I}$  such that  $g_k = h_k g_{k-1} h_k^{-1}$ . By §2.6 (b), there exists  $h_{n+1} \in \mathcal{I}\dot{s}\mathcal{I}$  such that  $h_{n+1} g_n h_{n+1}^{-1} \in \mathcal{I}\dot{w}_{n+1}\mathcal{I}$  with  $w_{n+1} \in \{sw_n, sw_n s\}$ . Now the statement follows from inductive hypothesis on  $w_{n+1}$ .  $\square$

**2.7. Proof of Proposition 2.5.** Let  $N_0 = \#\{w' \in W_a(M); \ell_M(w') \leq \ell_M(w)\}$ . By Lemma 2.7 and remark 2.8, there exists a standard triple  $(x, K, u)$  of  $\tilde{W}(M)$  and an element  $h \in \cup_{z \in W_a(M); \ell(z) \leq N_0} \mathcal{I}_M \dot{z} \mathcal{I}_M$  such that  $ux \in \tilde{W}(M)_{\min}$ ,  $w \rightarrow ux$  and  $h m h^{-1} \in \mathcal{I}_M \dot{u} \mathcal{I}_M$ .

Let  $i$  be a positive integer with  $iv \in X_*(Z)$ . Then  $x^i = t^{iv} \in \tilde{W}$  represents a central element in  $M$ . By §2.6 (c), for any  $l \in \mathbb{N}$ ,

$$(h m h^{-1})^{li} \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M t^{liv} \mathcal{I}_M).$$

Let  $N_1 = \max_{K \subset \tilde{\mathbb{S}}(M); W_K \text{ is finite}} \#W_K$ . Let  $i_{v,w} = (2N_0 + N_1 + 1)i$ . Then for any  $\alpha \in \Phi_{v,+}$ ,  $\langle i_{v,w} v, \alpha \rangle \geq 2N_0 + N_1 + 1$ . Note that  $m^{i_{v,w}} = h^{-1}(g_1 g_2)h$  with  $h \in \cup_{w' \in \tilde{W}(M); \ell(w') \leq N_0} \mathcal{I}_M \dot{w}' \mathcal{I}_M$ ,  $g_1 \in \cup_{u' \in \tilde{W}(M); \ell_M(u') \leq N_1} \mathcal{I}_M \dot{u}' \mathcal{I}_M$  and  $g_2 \in \mathcal{I}_M t^{i_{v,w} v} \mathcal{I}_M$ . So

$$\begin{aligned} m^{i_{v,w}} N_{v,n} (m^{i_{v,w}})^{-1} &= h^{-1} g_1 g_2 h N_{v,n} h^{-1} g_2^{-1} g_1^{-1} h \\ &\subset h^{-1} g_1 g_2 N_{v,n-N_0} g_2^{-1} g_1^{-1} h \\ &\subset h^{-1} g_1 N_{v,n-N_0+(2N_0+N_1+1)} g_1^{-1} h \\ &\subset h^{-1} N_{v,n-N_0+(2N_0+N_1+1)-N_1} h \\ &\subset N_{v,v,n-N_0+(2N_0+N_1+1)-N_1-N_0} = N_{v,n+1}. \end{aligned}$$

Similarly,  $m^{-i_{v,w}} N_{v,n}^- m^{i_{v,w}} \subset N_{v,n+1}^-$ .

### 3. THE MAP $\bar{i}_\nu$

We define the induction map  $\bar{i}_\nu$ , which is the main object in this paper.

**Theorem 3.1.** *Let  $M$  be a semistandard Levi subgroup of  $G$  and  $\nu \in \mathfrak{N}_M$  with  $M = M_\nu$ . Then*

(1) *For  $m \in M$  and an open compact subgroup  $\mathcal{K}_M$  of  $\mathcal{I}_M$  with  $m\mathcal{K}_M \subset M(\nu)$ , the map*

$$\delta_{m\mathcal{K}_M} \longmapsto \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M \mathcal{K})} \delta_{m\mathcal{K}_M \mathcal{K}} + [H, H]$$

from  $H(M; \nu)$  to  $\bar{H}(\bar{\nu})$  is independent the choice of sufficiently small open compact subgroup  $\mathcal{K}$  of  $G$

(2) The map  $i_\nu : H(M; \nu) \rightarrow \bar{H}$  defined above induces a map

$$\bar{i}_\nu : \bar{H}(M; \nu) \longrightarrow \bar{H}.$$

**Remark 3.2.** Unlike the map  $j_{v,n}$ , the map  $\bar{i}_\nu$  does not send  $\bar{H}(M, M_n; \nu)$  to  $\bar{H}(G, \mathcal{I}_n; \bar{\nu})$ . One needs to replace  $\mathcal{I}_n$  by a smaller open compact subgroup of  $G$ . However, by the Iwahori-Matsumoto presentation of  $\bar{H}(M, M_n; \nu)$  ([11, Theorem 4.1]) and Proposition 2.5, there exists a positive integer  $n'$  (depending on  $\nu$ ) such that  $\bar{i}_\nu : \bar{H}(M, M_n; \nu) \rightarrow \bar{H}(G, \mathcal{I}_{n+n'}; \bar{\nu})$  for any  $n \in \mathbb{N}$ .

*Proof.* (1) Let  $v$  be the  $V$ -factor of  $\nu$ . Let  $w \in \tilde{W}(M)$  with  $m \in \mathcal{I}_M w \mathcal{I}_M$ . Let  $i_{v,w}$  be an positive integer in Proposition 2.5. Let  $l$  be a multiple of  $i_{v,w} \ell(w)$  with  $M_l \subset \mathcal{K}_M$ . By Proposition 2.3 (2), for any  $n \in \mathbb{N}$  and  $g \in \mathcal{I}_l$ , we have

$$\delta_{m'gM_l\mathcal{I}_{l+n}} \equiv \delta_{m'M_l\mathcal{I}_{l+n}} \pmod{[H, H]}.$$

Let  $\mathcal{K}, \mathcal{K}'$  be open compact subgroups of  $G$  with  $\mathcal{K}, \mathcal{K}' \subset \mathcal{I}_l$ . Let  $n \in \mathbb{N}$  with  $\mathcal{I}_{l+n} \subset \mathcal{K}, \mathcal{K}'$ . Now we have

$$\begin{aligned} \delta_{m\mathcal{K}_M\mathcal{K}} &= \sum_{m' \in m\mathcal{K}_M/M_l} \delta_{m'M_l\mathcal{K}} \equiv \sum_{m' \in m\mathcal{K}_M/M_l} \frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})} \delta_{m'M_l\mathcal{I}_{l+n}} \\ &= \frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})} \delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \pmod{[H, H]}. \end{aligned}$$

As  $\mathcal{K}_M$  is stable under the right multiplication of  $M_l$ , we have  $\mu_G(\mathcal{K}_M\mathcal{I}_{l+n}) = \#(\mathcal{K}_M/M_l)\mu_G(M_l\mathcal{I}_{l+n})$  and  $\frac{\mu_G(M_l\mathcal{K})}{\mu_G(M_l\mathcal{I}_{l+n})} = \frac{\mu_G(\mathcal{K}_M\mathcal{K})}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})}$ . Thus for any  $n \in \mathbb{N}$ , we have

$$\frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})} \delta_{m\mathcal{K}_M\mathcal{K}} \equiv \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})} \delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \pmod{[H, H]}.$$

Similarly,  $\frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K}')} \delta_{m\mathcal{K}_M\mathcal{K}'} \equiv \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{I}_{l+n})} \delta_{m\mathcal{K}_M\mathcal{I}_{l+n}} \pmod{[H, H]}$ . Part (1) is proved.

(2) By [11, §3.3 (2)],  $[H(M), H(M)] = \bigoplus_{\nu \in \mathbb{N}_M} ([H(M), H(M)] \cap H(M)_\nu)$ , the kernel of the map  $H(M)_\nu \rightarrow \bar{H}(M)_\nu$  is spanned by  $\delta_{m\mathcal{K}_M} - {}^h\delta_{m\mathcal{K}_M}$  for  $h, m \in M$  and open compact subgroup  $\mathcal{K}_M$  of  $\mathcal{I}_M$  such that  $m\mathcal{K}_M \subset M_\nu$ . It remains to prove that  $i_\nu(\delta_{m\mathcal{K}_M}) = i_\nu({}^h\delta_{m\mathcal{K}_M})$ .

Set  $m' = h m h^{-1}$  and  $\mathcal{K}'_M = h\mathcal{K}_M h^{-1}$ . By part (1), there exists a sufficiently small open compact subgroup  $\mathcal{K}$  of  $G$  such that

$$\begin{aligned} i_\nu(\delta_{m\mathcal{K}_M}) &\equiv \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})} \delta_{m\mathcal{K}_M\mathcal{K}} \pmod{[H, H]}, \\ i_\nu(\delta_{m'\mathcal{K}'_M}) &\equiv \delta_\nu(m')^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}'_M)}{\mu_G(\mathcal{K}'_M\mathcal{K}')} \delta_{m'\mathcal{K}'_M\mathcal{K}'} \pmod{[H, H]}. \end{aligned}$$

Here  $\mathcal{K}' = h\mathcal{K}h^{-1}$ .

We have  $\delta_{m'\mathcal{K}'_M\mathcal{K}'} = \delta_{h(m\mathcal{K}_M\mathcal{K})h^{-1}} \equiv \delta_{m\mathcal{K}_M\mathcal{K}} \pmod{[H, H]}$ . Part (2) is proved.  $\square$

3.1. In the rest of this section, we show that the maps  $\bar{i}_*$  are compatible with conjugating the Levi subgroups.

For any semistandard Levi subgroup  $M$ , we have a natural projection

$$X_*(Z)_{\text{Gal}(\bar{F}/F)}/\mathbb{Z}\Phi_M^\vee \cong \Omega_M$$

and a natural map  $V \mapsto V_+^M$ . The natural action of  $W_0$  on  $X_*(Z)_{\text{Gal}(\bar{F}/F)} \times V$  induces the following commutative diagram for any  $w \in W_0$

$$\begin{array}{ccc} X_*(Z)_{\text{Gal}(\bar{F}/F)} \times V & \xrightarrow{w \cdot} & X_*(Z)_{\text{Gal}(\bar{F}/F)} \times V \\ \downarrow & & \downarrow \\ \mathfrak{N}_M & \xrightarrow{\quad} & \mathfrak{N}_{\dot{w}M\dot{w}^{-1}}. \end{array}$$

We denote the induced map  $\mathfrak{N}_M \rightarrow \mathfrak{N}_{\dot{w}M\dot{w}^{-1}}$  still by  $w \cdot$ . If moreover,  $w \in W^M$ , i.e.  $w$  sends the positive roots of  $M$  to the positive roots of  $\dot{w}M\dot{w}^{-1}$ , then we have  $\dot{w}\mathcal{I}_M\dot{w}^{-1} = \mathcal{I}_{\dot{w}M\dot{w}^{-1}}$ . By definition, the  $M$ -fundamental alcove is the unique  $M$ -alcove that contains the  $G$ -fundamental alcove. Since the conjugation by  $\dot{w}$  sends the Iwahori-subgroup of  $M$  to the Iwahori-subgroup of  $\dot{w}M\dot{w}^{-1}$ , it also sends the  $M$ -fundamental alcove to the  $\dot{w}M\dot{w}^{-1}$ -fundamental alcove, and thus induces a length-preserving map from  $\tilde{W}(M)$  to  $\tilde{W}(\dot{w}M\dot{w}^{-1})$ . In particular, the conjugation by  $w$  sends the minimal length elements of  $\tilde{W}(M)$  (with respect to  $\ell_M$ ) to the minimal length elements of  $\tilde{W}(\dot{w}M\dot{w}^{-1})$  (with respect to  $\ell_{\dot{w}M\dot{w}^{-1}}$ ). Therefore, by the definition of Newton strata, we have that

(a) Let  $M$  be a semistandard Levi subgroup  $M$  and  $\nu \in \mathfrak{N}_M$ . Let  $w \in W_0$  and  $M' = \dot{w}M\dot{w}^{-1}$ , then

$$\dot{w}M(\nu)\dot{w}^{-1} = M'(w(\nu)).$$

**Proposition 3.3.** *Let  $M$  be a semistandard Levi subgroup and  $\nu \in \mathfrak{N}_M$  and  $w \in W_0$ . Then for any  $m \in M$ , and an open compact subgroup  $\mathcal{K}_M$  of  $\mathcal{I}_M$  with  $m\mathcal{K}_M \subset M_\nu$  and  $\dot{w}\mathcal{K}_M\dot{w}^{-1} \subset \mathcal{I}_{\dot{w}M\dot{w}^{-1}}$ , we have*

$$i_\nu(\delta_{m\mathcal{K}_M}) = i_{w(\nu)}(\delta_{\dot{w}m\mathcal{K}_M\dot{w}^{-1}}) \in \bar{H}.$$

*Proof.* The proof is similar to the proof of Theorem 3.1 (2).

Set  $M' = \dot{w}M\dot{w}^{-1}$ ,  $m' = \dot{w}m\dot{w}^{-1}$  and  $\mathcal{K}_{M'} = \dot{w}\mathcal{K}_M\dot{w}^{-1}$ . By Theorem 3.1 (1), there exists a sufficiently small open compact subgroup  $\mathcal{K}$  of  $G$  such that

$$i_\nu(\delta_{m\mathcal{K}_M}) \equiv \delta_\nu(m)^{-\frac{1}{2}} \frac{\mu_M(\mathcal{K}_M)}{\mu_G(\mathcal{K}_M\mathcal{K})} \delta_{m\mathcal{K}_M\mathcal{K}} \pmod{[H, H]},$$

$$i_{w(\nu)}(\delta_{m'\mathcal{K}_{M'}}) \equiv \delta_{w(\nu)}(m')^{-\frac{1}{2}} \frac{\mu_{M'}(\mathcal{K}_{M'})}{\mu_G(\mathcal{K}_{M'}\mathcal{K}')} \delta_{m'\mathcal{K}_{M'}\mathcal{K}'} \pmod{[H, H]}.$$

Here  $\mathcal{K}' = \dot{w}\mathcal{K}\dot{w}^{-1}$ .

We have  $\delta_{m'\mathcal{K}_{M'}\mathcal{K}'} = \delta_{\dot{w}(m\mathcal{K}_M\mathcal{K})\dot{w}^{-1}} \equiv \delta_{m\mathcal{K}_M\mathcal{K}} \pmod{[H, H]}$ . The statement is proved.  $\square$

**Corollary 3.4.** *Let  $M$  be a semistandard Levi subgroup of  $G$  and  $\nu \in \mathfrak{N}_M$  with  $M = M_\nu$ . Then for any  $w \in W_0$ ,*

$$\text{Im}(\bar{i}_\nu : \bar{H}(M; \nu) \longrightarrow \bar{H}) = \text{Im}(\bar{i}_{w(\nu)} : \bar{H}(\dot{w}M\dot{w}^{-1}; w(\nu)) \longrightarrow \bar{H}).$$

#### 4. THE IMAGE OF THE MAP $\bar{i}_\nu$

The main result of this section is

**Theorem 4.1.** *Let  $M$  be a semistandard Levi subgroup and  $\nu \in \mathfrak{N}_M$  with  $M = M_\nu$ . Then the image of the map  $\bar{i}_\nu : \bar{H}(M; \nu) \rightarrow \bar{H}$  equals  $\bar{H}(\bar{\nu})$ .*

We first compare the Newton strata of  $G$  and its Levi subgroups.

**Proposition 4.2.** *Let  $M$  be a semistandard Levi subgroup and  $\nu \in \mathfrak{N}_M$  with  $M_\nu = M$ . Then we have  $M(\nu) \subset G(\bar{\nu})$ .*

*Proof.* The idea is similar to the proof of [11, Theorem 2.1].

By §3.1 (a), after conjugating by a suitable element in  $W_0$ , we may assume that  $M$  is a standard Levi subgroup. Since  $M = M_\nu$ , the  $V$ -factor of  $\nu$  is  $G$ -dominant. By the Newton decomposition of  $G$  ([11, Theorem 2.1]), it suffices to prove that  $M(\nu) \cap G(\nu') = \emptyset$  for any  $\nu' \in \mathfrak{N}$  with  $\nu' \neq \bar{\nu}$ .

Let  $\nu = (\tau, v)$  and  $\nu' = (\tau', v')$ . If the image of  $\tau$  in  $\Omega$  does not equal to  $\tau'$ , then  $M(\nu) \cap G(\nu') = \emptyset$ . Now we assume that the  $\Omega$ -factor matches. Since  $\nu' \neq \bar{\nu}$ , we have  $v' \neq v$ .

By [11, Remark 2.6],

$$M(\nu) = \cup_{(x, K, u)} M \cdot \mathcal{I}_M \dot{u} x \mathcal{I}_M, \quad G(\nu') = \cup_{(x', K', u')} G \cdot \mathcal{I} \dot{u}' x' \mathcal{I},$$

where  $(x, K, u)$  runs over standard triples of  $\tilde{W}(M)$  such that  $ux \in \tilde{W}(M)_{\min}$  and  $\pi_M(x) = \nu$ ,  $(x', K', u')$  runs over standard triples of  $\tilde{W}$  such that  $u'x' \in \tilde{W}_{\min}$  and  $\pi(x') = \nu'$ .

If  $M(\nu) \cap G(\nu') \neq \emptyset$ , then there exists standard triples  $(x, K, u)$  and  $(x', K', u')$  as above and  $h \in \mathcal{I}_M \dot{u} x \mathcal{I}_M, h' \in \mathcal{I} \dot{u}' x' \mathcal{I}, g \in G$  such that  $ghg^{-1} = h'$ . For any  $n \in \mathbb{N}$ , we have  $gh^n g^{-1} = (h')^n$ . By §2.6 (c), we have

$$h^n \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M \dot{x}^n \mathcal{I}_M), \quad (h')^n \in (\mathcal{I} W_{K'} \mathcal{I})(\mathcal{I} \dot{x}'^n \mathcal{I}).$$

Let  $l > 0$  with  $lv, lv' \in X_*(Z)$ . Suppose that  $g \in \mathcal{I} z \mathcal{I}$  for some  $z \in \tilde{W}$ . Then for any  $n \in \mathbb{N}$ , we have

$$\mathcal{I} z \mathcal{I} t^{nlv} \mathcal{I} (\mathcal{I} W_K \mathcal{I}) \mathcal{I} z^{-1} \mathcal{I} (\mathcal{I} W_{K'} \mathcal{I}) \cap \mathcal{I} t^{nlv'} \mathcal{I} \neq \emptyset.$$

Similar to the argument in [11, §2.6], this is impossible for  $n \gg 0$ . The statement is proved.  $\square$

**Corollary 4.3.** *The image of the map  $\bar{i}_\nu$  is contained in  $\bar{H}(\bar{\nu})$ .*

*Proof.* Let  $m \in M$  and  $\mathcal{K}_M$  be an open compact subgroup of  $\mathcal{I}_M$  with  $m\mathcal{K}_M \subset M(\nu)$ . By Proposition 4.2,  $m\mathcal{K}_M \subset G(\bar{\nu})$ . Let  $X$  be an open compact subset of  $G$  with  $m\mathcal{K}_M \subset X$ . By [11, Theorem 3.2], there exists  $n \in \mathbb{N}$  such that  $X \cap G(\bar{\nu})$  is stable under the right multiplication by  $\mathcal{I}_n$ . In particular,  $m\mathcal{K}_M \mathcal{I}_n \subset G(\bar{\nu})$ . Thus  $\bar{i}_\nu(\delta_{m\mathcal{K}_M}) \in \bar{H}(\bar{\nu})$ .  $\square$

4.1. In order to prove the other direction, we use the notion of alcove elements in [8] and [9].

Let  $w \in \tilde{W}$ . We may regard  $w \in \text{Aff}(V)$  as an affine transformation. Let  $p : \text{Aff}(V) = V \rtimes GL(V) \rightarrow GL(V)$  be the natural projection map. Let  $v \in V$ . We say that  $w$  is a  $v$ -alcove element if

- $p(w)(v) = v$ ;
- $N_v \cap \dot{w} \mathcal{I} w^{-1} \subset N_v \cap \mathcal{I}$ .

Note that the first condition implies that  $\dot{w} M_v \dot{w}^{-1} = M_v$ . We have the following result.

**Theorem 4.4.** *Let  $w \in \tilde{W}$ . If  $w$  is a  $\nu_w$ -alcove element, then any element in  $\mathcal{I} \dot{w} \mathcal{I}$  is conjugate by  $\mathcal{I}$  to an element in  $\dot{w} \mathcal{I}_{M_{\nu_w}}$ .*



*Proof.* The basic idea is similar to the proof of [8, Theorem 2.1.2].

Write  $M$  for  $M_{\nu_w}$  and  $N$  for  $N_{\nu_w}$ . We start with the generic Moy-Prasad filtration  $\mathcal{I} = \mathcal{I}[0] \supset \mathcal{I}[1] \supset \dots$ . As explained in [8, §6.2], it is a filtration satisfying the following conditions:

- (1) Each  $\mathcal{I}[r]$  is normal in  $\mathcal{I}$ ;
- (2) For each  $r$ , either  $\mathcal{I}[r] \subset \mathcal{I}_M \mathcal{I}[r+1]$  or there exists a root  $a \in \Phi - \Phi(M)$  and  $s \in \mathbb{R}$  such that  $\mathcal{I}[r] = X_{a+s} \mathcal{I}[r+1]$  and  $X_{a+s+\epsilon} \subset \mathcal{I}[r+1]$  for any  $\epsilon > 0$ .

We show that each element  $\dot{w}i_M i[r]$  with  $i_M \in \mathcal{I}_M$  and  $i[r] \in \mathcal{I}[r]$  is conjugate by an element in  $\mathcal{I}$  to an element in  $\dot{w}\mathcal{I}_M \mathcal{I}[r+1]$  (and that the conjugator can be taken to be small when  $r$  is large).

If  $\mathcal{I}[r] \subset \mathcal{I}_M \mathcal{I}[r+1]$ , then we may absorb the  $\mathcal{I}_M$  part into  $i_M$ . Otherwise, there exists a root  $a$  outside  $M$  such that  $\mathcal{I}[r] = X_{a+s} \mathcal{I}[r+1]$  and  $X_{a+s+\epsilon} \subset \mathcal{I}[r+1]$  for any  $\epsilon > 0$ . We prove the case where  $a$  is a root in  $N$ . The case where  $a$  is a root in  $N^-$  can be proved in the same way.

We have  $i[r] \in u\mathcal{I}[r+1]$  for some  $u \in X_{a+s} \subset N_s$ . Set  $m = \dot{w}i_M$ . By the definition of  $P$ -alcove elements,  $m^i u (m^i)^{-1} \subset N_s$  for all  $i \in \mathbb{N}$ . As in the proof of Proposition 2.3 (1),  $\dot{w}i_M i[r]$  is conjugate by elements in  $N_s$  to elements in

$$\begin{aligned} mu\mathcal{I}[r+1] &= (mum^{-1})m\mathcal{I}[r+1] \sim m\mathcal{I}[r+1](mum^{-1}) = m(mum^{-1})\mathcal{I}[r+1] \\ &= (m^2u(m^2)^{-1})m\mathcal{I}[r+1] \sim \dots \sim (m^i u (m^i)^{-1})m\mathcal{I}[r+1] \sim \dots \end{aligned}$$

Here  $\sim$  means conjugation by elements in  $N_s$ .

By Proposition 2.5, there exist  $i \in \mathbb{N}$  such that  $m^i u (m^i)^{-1} \subset \mathcal{I}[r+1]$ . Thus  $\dot{w}i_M i[r]$  is conjugate by an element in  $N_s$  to an element in  $\dot{w}\mathcal{I}_M \mathcal{I}[r+1]$ .

Now we start with an element in  $\dot{w}\mathcal{I}$ . The convergent product of the conjugators (for all  $r$ ) is an element in  $\mathcal{I}$  and conjugates the given element to an element in  $\dot{w}\mathcal{I}_M$ .  $\square$

**4.2. Proof of Theorem 4.1.** By Corollary 4.3, the image of  $\bar{i}_\nu$  is contained in  $\bar{H}(\bar{\nu})$ . Now we prove the other direction. By [11, Corollary 4.2],

$$\bar{H} = \sum_{w \in \tilde{W}_{\min}; \pi(w) = \bar{\nu}} \bar{H}_w,$$

where  $H_w$  is the submodule of  $H$  consisting of functions supported in  $\mathcal{I}\dot{w}\mathcal{I}$  and  $\bar{H}_w$  is the image of  $H_w$  in  $\bar{H}$ .

Let  $w \in \tilde{W}_{\min}$  with  $\pi(w) = \bar{\nu}$ . By [9, Lemma 4.4.3 and Proposition 4.4.6],  $w$  is a  $\nu_w$ -alcove element. Set  $M' = M_{\nu_w}$  and  $\nu' = \pi_{M'}(w) \in \mathfrak{N}_{M'}$ .

Let  $i_{\nu', w}$  be a positive integer in Proposition 2.5. By definition,  $H_w$  is spanned by  $\delta_{g\mathcal{I}_n}$  for  $g \in \mathcal{I}\dot{w}\mathcal{I}$  and  $n > i(\nu', w)\ell(w)$ . By the proof of Theorem 3.1 (1), for any  $n > i(\nu', w)\ell(w)$  and  $g \in \dot{w}\mathcal{I}_{M'}$ ,  $\delta_{g\mathcal{I}_n} + [H, H]$  is contained in the image of  $\bar{i}_{\nu'}$ .

Let  $g \in \mathcal{I}\dot{w}\mathcal{I}$ . By Theorem 4.4, there exists  $i \in \mathcal{I}$  and  $g' \in \dot{w}\mathcal{I}_{M'}$  such that  $g = ig'i^{-1}$ . Then

$$\delta_{g\mathcal{I}_n} = \delta_{ig'\mathcal{I}_n i^{-1}} \equiv \delta_{g'\mathcal{I}_n} \pmod{[H, H]}.$$

Therefore  $\bar{H}_w$  is contained in the image of  $\bar{i}_{\nu'}$ . By Proposition 3.3,  $\bar{H}_w$  is also contained in the image of  $\bar{i}_\nu$ .

## 5. ADJUNCTION WITH THE JACQUET FUNCTOR

5.1. Let  $R$  be an algebraically closed field of characteristic  $\neq p$ . Set  $H_R = H \otimes_{\mathbb{Z}[\frac{1}{p}]} R$ ,  $\bar{H}_R = \bar{H} \otimes_{\mathbb{Z}[\frac{1}{p}]} R$  and  $\bar{H}_R(\nu) = \bar{H}(\nu) \otimes_{\mathbb{Z}[\frac{1}{p}]} R$ . Recall that  $\mathfrak{R}(G)_R$  is the  $R$ -vector space with basis the isomorphism classes of irreducible smooth admissible representations of  $G$  over  $R$ . We consider the trace map

$$\mathrm{Tr}_R^G : \bar{H}_R \longrightarrow \mathfrak{R}(G)_R^*.$$

Similarly, for any semistandard Levi subgroup  $M$ , we have

$$\mathrm{Tr}_R^M : \bar{H}_R(M) \longrightarrow \mathfrak{R}(M)_R^*.$$

Let  $v \in V$  and  $M = M_v$ . Let  $r_{v,R} : \mathfrak{R}(G)_R \rightarrow \mathfrak{R}(M)_R$  be the (normalized) Jacquet functor. Note that the Jacquet functor does not only depend on the Levi  $M$ , but also depends on the direction  $v$  (or equivalently, the parabolic subgroup  $P_v$  with Levi factor  $M$ ). The following result is proved by Bushnell in [2, Corollary 1].

**Proposition 5.1.** *Let  $n \in \mathbb{N}$ . Let  $v \in V$  and  $M = M_v$ . Then for any  $f \in H_R^v(M, M_n)$ , and  $\pi \in \mathfrak{R}_{\mathcal{I}_n}(G)_R$ , we have*

$$\mathrm{Tr}_R^M(f, r_{v,R}(\pi)) = \mathrm{Tr}_R^G(j_{v,n}(f), \pi).$$

The main result of this section is the following adjunction formula.

**Theorem 5.2.** *Let  $M$  be a semistandard Levi subgroup and  $\nu \in \mathfrak{N}_M$ . Suppose that  $M = M_\nu$ . Then for any  $f \in \bar{H}_R(M; \nu)$  and  $\pi \in \mathfrak{R}(G)_R$ , we have*

$$\mathrm{Tr}_R^M(f, r_{\nu,R}(\pi)) = \mathrm{Tr}_R^G(\bar{i}_\nu(f), \pi).$$

5.2. Let  $(x, K, u)$  be a standard triple of  $\tilde{W}(M)$  such that the Newton point of  $x$  is  $v$ . Let  $\mathbf{i}$  be the smallest positive integer with  $\mathbf{i}v \in X_*(Z)$ . Let  $i \in \mathbb{N}$  such that for any  $\alpha \in \Phi_{v,+}$ ,  $\langle \mathbf{i}v, \alpha \rangle \geq \sharp W_K + (\mathbf{i} - 1)\ell(x) + 1$ . Let  $l \geq \mathbf{i}i$ . Then  $l = i'\mathbf{i} + j$  for some  $i' \geq i$  and  $0 \leq j < \mathbf{i}$ . Then for any  $m_1, \dots, m_l \in \mathcal{I}_M \dot{u} \dot{x} \mathcal{I}_M$ , by §2.6 (c), we have

$$m_1 m_2 \cdots m_l \in (\mathcal{I}_M W_K \mathcal{I}_M)(\mathcal{I}_M \dot{x}^j \mathcal{I}_M)(\mathcal{I}_M t^{i'v} \mathcal{I}_M).$$

Note that for  $g \in \mathcal{I} t^{i'v} \mathcal{I}$ ,  $g N_n g^{-1} \subset N_{n+\sharp W_K + (\mathbf{i}-1)\ell(x)+1}$ . Also  $(\mathcal{I} W_K \mathcal{I})(\mathcal{I} \dot{x}^j \mathcal{I}) \subset \cup_{w \in \tilde{W}; \ell(w) \leq \sharp W_K + (\mathbf{i}-1)\ell(x)} \mathcal{I} \dot{w} \mathcal{I}$ . Thus  $(m_1 \cdots m_l) N_n (m_1 \cdots m_l)^{-1} \subset N_{n+1}$ . Similarly  $(m_1 \cdots m_l)^{-1} N_n^- (m_1 \cdots m_l) \subset N_{n+1}^-$ . Therefore,

(a) Let  $l \geq \mathbf{i}i$  and  $m_1, \dots, m_l \in \mathcal{I}_M \dot{u} \dot{x} \mathcal{I}_M$ , then  $m_1 m_2 \cdots m_l$  is a  $P_v$  strictly positive element.

Moreover, for any  $n, l' \in \mathbb{N}$  and  $m_1, \dots, m_{l'} \in \mathcal{I}_M \dot{u} \dot{x} \mathcal{I}_M$ , we have

$$\begin{aligned} (m_1 \cdots m_{l'}) N_{n+\sharp W_K + (\mathbf{i}-1)\ell(x)} (m_1 \cdots m_{l'})^{-1} &\subset N_n, \\ (m_1 \cdots m_{l'})^{-1} N_{n+\sharp W_K + (\mathbf{i}-1)\ell(x)}^- (m_1 \cdots m_{l'}) &\subset N_n^-. \end{aligned}$$

One deduces that

(b) Let  $n, l' \in \mathbb{N}$ , and  $g_1, \dots, g_{l'} \in N_{n+\sharp W_K + (\mathbf{i}-1)\ell(x)} \mathcal{I}_M \dot{u} \dot{x} \mathcal{I}_M N_{n+\sharp W_K + (\mathbf{i}-1)\ell(x)}^-$ . Then  $g_1 \cdots g_{l'} \in N_n M N_n^-$ .

**5.3. Proof of Theorem 5.2.** By [11, Theorem 4.1 & §4.6], it suffices to prove it for locally constant functions on  $M$ , supported in  $M\dot{u}xM$ , where  $(x, K, u)$  is a standard triple of  $\tilde{W}(M)$  and the Newton point of  $x$  is  $v$ .

Let  $n > \sharp W_K + (\mathfrak{i} - 1)\ell(x)$  such that  $\pi \in \mathfrak{R}_{\mathcal{L}_n}(G)_R$ . It is enough to consider the function  $f = \delta_{M_n m M_n}$ , where  $m \in M\dot{u}xM$ .

Let  $n' \gg n$  and  $\tilde{f} = \frac{\delta_v(m)^{-\frac{1}{2}}}{\mu_N(N_{n'})\mu_{N^-}(N_{n'}^-)} \delta_{N_{n'} M_n m M_n N_{n'}^-}$ . By Theorem 3.1 (1),  $\tilde{f}$  represents the element  $\bar{i}_v(f) \in \bar{H}$ . By Casselman's trick [4, Corollary 4.2], it suffices to prove that for  $l \gg 0$ ,  $\text{Tr}_R^M(f^l, r_v(\pi)) = \text{Tr}_R^G(\tilde{f}^l, \pi)$ .

Let  $p_M : (M_n m M_n)^l \rightarrow M$  and  $p_G : (N_{n'} M_n m M_n N_{n'}^-)^l \rightarrow G$  be the multiplication map. Since  $l \gg 0$ , by §5.2 (a) and (b), any element in  $\text{Im}(p_M)$  is  $P_\nu$  strictly positive and

$$\text{Im}(p_G) \subset N_n \text{Im}(p_M) N_n^- \cong N \times \text{Im}(p_M) \times N^-.$$

We have the following commutative diagram

$$\begin{array}{ccc} (N_{n'} M_n m M_n N_{n'}^-)^l & \xrightarrow{p_G} & \text{Im}(p_G) \\ \downarrow pr^l & & \downarrow pr_1 \\ (M_n m M_n)^l & \xrightarrow{p_M} & \text{Im}(p_M), \end{array}$$

where  $pr : N \times M \times N^- \rightarrow M$  is the projection map and  $pr_1$  is the restriction of  $pr$  to  $\text{Im}(p_G)$ .

Let  $m' \in \text{Im}(p_M)$ . Then

$$\begin{aligned} \mu_{G^i}(p_G^{-1} pr_1^{-1}(M_n m' M_n)) &= \mu_{G^i}((pr^l)^{-1} p_M^{-1}(M_n m' M_n)) \\ &= \mu_N(N_{n'})^l \mu_{N^-}(N_{n'}^-)^l \mu_{M^i}(p_M^{-1}(M_n m' M_n)). \end{aligned}$$

By Proposition 2.3 (2),  $\delta_{i\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'} i'} \equiv \delta_{\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}}$  mod  $[H, H]$  for any  $i \in N_n$  and  $i' \in N_{n'}^-$ . Thus

$$\begin{aligned} \tilde{f}^l &\equiv \frac{\delta_v(m)^{-\frac{1}{2}}}{\mu_N(N_{n'})^l \mu_{N^-}(N_{n'}^-)^l} \sum_{m' \in M_n \backslash M / M_n} \frac{\mu_{G^i}(p_G^{-1} pr_1^{-1}(M_n m' M_n))}{\mu_G(pr_1^{-1}(M_n m M_n))} \delta_{pr_1^{-1}(M_n m M_n)} \\ &\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \frac{\mu_{M^i}(p_M^{-1}(M_n m' M_n))}{\mu_G(pr_1^{-1}(M_n m M_n))} \delta_{pr_1^{-1}(M_n m M_n)} \\ &\equiv \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \frac{\mu_{M^i}(p_M^{-1}(M_n m' M_n))}{\mu_G(\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'})} \delta_{\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}} \quad \text{mod } [H, H]. \end{aligned}$$

On the other hand,

$$f^l = \sum_{m' \in M_n \backslash M / M_n} \frac{\mu_{M^i}(p_M^{-1}(M_n m' M_n))}{\mu_M(M_n m' M_n)} \delta_{M_n m' M_n}.$$

By Corollary 2.4, we have

$$\begin{aligned} j_{v,n}(f^l) &\equiv j_{v,n'}(f^l) \\ &= \sum_{m' \in M_n \backslash M / M_n} \delta_v(m)^{-\frac{1}{2}} \frac{\mu_{M^i}(p_M^{-1}(M_n m' M_n))}{\mu_M(M_n m' M_n)} \frac{\mu_M(M_{n'})}{\mu_G(\mathcal{I}_{n'})} \delta_{\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}} \quad \text{mod } [H, H]. \end{aligned}$$

Since the elements in  $M_n m' M_n$  are  $P_v$  strictly positive, we have  $\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'} = N_{n'}(M_n m' M_n) N_{n'}^-$  and

$$\mu_G(\mathcal{I}_{n'} M_n m' M_n \mathcal{I}_{n'}) = \mu_N(N_{n'}) \mu_{N^-}(N_{n'}^-) \mu_M(M_n m' M_n) = \frac{\mu_G(\mathcal{I}_{n'})}{\mu_M(M_{n'})} \mu_M(M_n m' M_n).$$

So  $\tilde{f}^l \equiv j_{v,n}(f^l) \pmod{[H, H]}$  and  $\mathrm{Tr}_R^M(f^l, r_v(\pi)) = \mathrm{Tr}_R^G(\tilde{f}^l, \pi)$ .

## 6. THE KERNEL OF THE TRACE MAP

6.1. Let  $M$  be a semistandard Levi subgroup of  $G$ . Let  $M^0$  be the subgroup of  $G$  generated by the parahoric subgroups of  $M$ . Then we have  $M/M^0 \cong \Omega_M$ . Let  $\Psi(M)_R = \mathrm{Hom}_{\mathbb{Z}}(M/M^0, R^\times)$  be the torus of unramified characters of  $M$ .

Let  $i_{M,R} : \mathfrak{R}(M)_R \rightarrow \mathfrak{R}(G)_R$  be the induction functor. Then for any  $\sigma \in \mathfrak{R}(M)_R$  and  $f \in \bar{H}_R$ , the map

$$\Psi(M)_R \longrightarrow R, \quad \chi \longmapsto \mathrm{Tr}_R(f, i_{M,R}(\sigma \circ \chi))$$

is an algebraic function over  $\Psi(M)_R$ .

6.2. Let  $v \in V$  and  $M = M_v$ . Recall that

$$\begin{aligned} \text{(a)} \quad & \bar{H}(M; v) = \bigoplus_{\nu_M \in \mathfrak{N}_M; \nu = (\tau_M, v) \text{ for some } \tau_M \in \Omega_M} \bar{H}(M; \nu), \\ \text{(b)} \quad & \bar{H}(\bar{v}) = \bigoplus_{\nu \in \mathfrak{N}; \nu = (\tau, \bar{v}) \text{ for some } \tau \in \Omega} \bar{H}(\nu). \end{aligned}$$

Note that if  $\tau_M, \tau'_M \in \Omega_M$  are mapped under  $\kappa$  to the same element in  $\Omega$ , then they differ by a central cocharacter of  $M$ . By the definition of the map  $\pi = (\kappa, \bar{\nu})$ , if both  $(\tau_M, v)$  and  $(\tau'_M, v)$  are in the image of  $\pi_M$  and that  $\kappa(\tau_M) = \kappa(\tau'_M)$ , then  $\tau_M = \tau'_M$ . In other words, there is a natural bijection between the components appear on the right hand sides of (a) and (b). We define

$$\bar{i}_v = \bigoplus_{\nu_M \in \mathfrak{N}_M; \nu = (\tau_M, v) \text{ for some } \tau_M \in \Omega_M} \bar{i}_\nu : \bar{H}(M; v) \longrightarrow \bar{H}(\bar{v}).$$

**Theorem 6.1.** *Let  $v \in V$  and  $M = M_v$ . Let  $f \in \bar{H}(\bar{v})$ . If  $\mathrm{Tr}_R^G(f, i_{M,R}(\sigma)) = 0$  for all  $\sigma \in \mathfrak{R}(M)_R$ , then  $f \in \bar{i}_v(\ker \mathrm{Tr}_R^M)$ .*

*Proof.* For  $\sigma \in \mathfrak{R}(M)_R$  and  $\chi \in \Psi(M)_R$ , the map

$$\chi \longmapsto \mathrm{Tr}_R^G(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi))$$

is an algebraic function on  $\chi$ . We consider its “positive part”, i.e. the linear combination of the terms  $\langle \chi, \lambda \rangle$  for dominant coweight  $\lambda$ . It is obvious that if an algebraic function is zero, then its “positive part” is also zero.

By the Mackey formula [16, §5.5], we have

$$\begin{aligned} \mathrm{Tr}_R^G(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi)) &= \mathrm{Tr}_R^M(f, r_{M,R} \circ i_{M,R}(\sigma \circ \chi)) \\ &= \sum_{w \in {}^M W^M} \mathrm{Tr}_R^M(f, i_{M \cap {}^w M, R}^M \circ \dot{w} \circ r_{M \cap {}^{w^{-1}} M, R}^M(\sigma \circ \chi)) \\ &= \sum_{w \in {}^M W^M} \mathrm{Tr}_R^M(f, i_{M \cap {}^w M, R}^M(\dot{w} \circ r_{M \cap {}^{w^{-1}} M, R}^M(\sigma) \circ \dot{w} \chi)). \end{aligned}$$

As  $w \in {}^M W^M$  and  $M = M_v$ ,  $w(v)$  is dominant if and only if  $w = 1$ . Therefore the “positive part” of  $\mathrm{Tr}_R^G(\bar{i}_v(f), i_{M,R}(\sigma \circ \chi))$  is  $\mathrm{Tr}_R^M(f, \sigma \circ \chi)$ .

Therefore if  $\mathrm{Tr}_R^G(f, i_{M,R}(\sigma)) = 0$  for any  $\sigma \in \mathfrak{R}(M)_R$  and  $\chi \in \Psi(M)_R$ , then  $\mathrm{Tr}_R^M(f, \sigma \circ \chi) = 0$  for any  $\sigma \in \mathfrak{R}(M)_R$  and  $\chi \in \Psi(M)_R$ . Hence  $f \in \ker \mathrm{Tr}_R^M$ .  $\square$

**Corollary 6.2.** *Let  $v \in V$  and  $M = M_v$ . Then*

$$\bar{i}_v^{-1}(\ker \mathrm{Tr}_R^G |_{\bar{H}_R(\bar{v})}) = \ker \mathrm{Tr}_R^M |_{\bar{H}_R(M;v)}.$$

*Proof.* If  $f \in \ker \mathrm{Tr}_R^M$ , then  $\mathrm{Tr}_R^M(f, r_{M,R}(\pi)) = 0$  for any  $\pi \in \mathfrak{R}(G)_R$ . By Theorem 5.2,  $\mathrm{Tr}_R^G(\bar{i}_v(f), \pi) = 0$ . Thus  $\bar{i}_v(f) \in \ker \mathrm{Tr}_R^G$ . The other direction follows from Theorem 6.1.  $\square$

**Theorem 6.3.** *We have  $\ker \mathrm{Tr}_R^G = \bigoplus_{v \in V_+} (\ker \mathrm{Tr}_R^G \cap \bar{H}_R(v))$ .*

**Remark 6.4.** In general,  $\bigoplus_{\nu \in \mathfrak{N}} (\ker \mathrm{Tr}_R^G \cap \bar{H}_R(\nu)) \subset \ker \mathrm{Tr}_R^G$ . However, the equality may not hold. For example, if  $\Omega = \{1, \tau\}$  is finite of order 2 and characteristic of  $R$  is also 2, then for any  $\lambda \in X_*(Z)_+$  and  $f \in \bar{H}(\lambda)$ , we have  $f + \tau f \in \ker \mathrm{Tr}_R^G$ .

*Proof.* The idea is similar to the proof of [5, Theorem 7.1].

Let  $f = \sum_{v \in V_+} a_v f_v \in \ker \mathrm{Tr}_R^G$ , where  $f_v \in \bar{H}_v$  and  $a_v \in R$ . Let  $M$  be a minimal standard Levi subgroup such that  $a_v \neq 0$  for some  $v \in V_+$  with  $M = M_v$ . Then for  $\sigma \in R(M)$  and  $\chi \in \Psi(M)_R$ , we have

$$(a) \quad \mathrm{Tr}_R^G(f, i_{M,R}(\sigma \circ \chi)) = \sum_{v \in V_+; M=M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) + \sum_{v \in V_+; M \neq M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)).$$

This is an algebraic function on  $\Psi(M)_R$ . Note that in (a), the first part is more regular in  $\Psi(M)_R$  than the second part. Therefore we have

$$\sum_{v \in V_+; M=M_v} a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) = 0$$

for all  $\sigma \in R(M)$  and  $\chi \in \Psi(M)_R$ . As an algebraic function on  $\Psi(M)_R$ , the “leading term” of  $\mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi))$  is a multiple of  $\langle v, \chi \rangle$ . Hence  $a_v \mathrm{Tr}_R^G(f_v, i_{M,R}(\sigma \circ \chi)) = 0$  for every  $v \in V_+$  with  $M = M_v$ . By Theorem 6.1,

$$a_v f_v \in \bar{i}_v(\ker \mathrm{Tr}_R^M |_{\bar{H}(M;v)}). \quad \square$$

Finally, we have

**Theorem 6.5.** *Assume that  $\mathrm{char}(F) = 0$ . Let  $M$  be a semistandard Levi subgroup and  $\nu \in \mathfrak{N}_M$  with  $M = M_\nu$ . Then the map*

$$\bar{i}_\nu : \bar{H}(M; \nu) \xrightarrow{\cong} \bar{H}(\bar{\nu})$$

*is an isomorphism.*

*Proof.* Let  $f \in \ker \bar{i}_\nu$ . Set  $\tilde{f} = f \otimes 1 \in \bar{H}_\mathbb{C}(M; \nu)$ . By Theorem 6.1 (2), we have  $\tilde{f} \in \ker \mathrm{Tr}_\mathbb{C}^M$ . By the spectral density theorem [14, Theorem 0],  $\tilde{f} = 0 \in \bar{H}(M)_\mathbb{C}$ . By [17],  $\bar{H}(M)$  is free. Hence  $f = 0 \in \bar{H}(M)$ .  $\square$

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